ROTATIONAL MOTIONS OF A SOLID BODY WITH A CAVITY FILLED WITH FLUID

PMM Vol. 31, No. 3, 1967, pp. 416-432

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(Received November 28, 1966)

It is well known that problems of dynamics of a rotating fluid have a series of specific peculiarities and present significant difficulties. In recent years equations of motion of a rotating fluid have been investigated in papers of Sobolev, R.A. Aleksandrian, S.G. Krein and others. Motion of a symmetrical top with a cavity filled with an ideal fluid was studied in the paper of [1] and subsequently (by a different method) in [2]. Theorems on stability of motion of a solid body with a cavity filled with fluid were proved by Rumiantsev and other authors (see book [3], which contains a bibliography). A number of papers, for example [4 to 6], is devoted to the analysis of motion of fluid in a cavity of a solid body executing a prescribed motion: uniform rotation or regular precession. The general problem of motion of a body with a cavity filled with a viscous fluid was examined in [7] for the case of high viscosity fluid and in [8] for the case of low viscosity under the condition that the body executes small oscillations.

In this paper the motion of a solid body with a fluid-filled cavity is examined under the following assumptions. Distribution of masses in the body and the shape of the cavity are considered arbitrary, the fluid is ideal or has low viscosity. The motion of the body with the fluid is assumed to be close to uniform rotation around an axis. One property of natural oscillations of the liquid rotating in the cavity is established. Special solutions of linearized equations of rotational motion of an ideal fluid are brought into the investigation. These solutions depend on the shape of the cavity and are analogous to Zhukovskii's potentials for the case of irrotational motion. It is shown that through these solutions the angular momentum of the system is expressed by means of some tensors in the case of an ideal fluid and also in case of a fluid with low viscosity. Some concrete shapes of cavities and particular cases of motion are also examined. The characteristic equation for oscillations of a rotating free solid body with a fluid-filled cavity is obtained and in some cases analyzed.

1. Basic equations. Let us examine the motion of a solid body with a cavity D filled with an incompressible fluid of density ρ and kinematic viscosity ν. Navier-Stokes equations and boundary conditions are written in the system of coordinates $Ox_1x_2x_3$, which is rigidly connected with the solid body

$$\mathbf{w}_0+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})+\boldsymbol{\omega}^{\cdot}\times\mathbf{r}+2\boldsymbol{\omega}\times\mathbf{V}+\partial\mathbf{V}/\partial t+(\mathbf{V}\nabla)\mathbf{V}=-\rho^{-1}\nabla P+\nabla U+\nu\Delta\mathbf{V}$$
$$\operatorname{div}\mathbf{V}=0 \quad \text{in } D, \qquad \mathbf{V}=0 \quad (\mathbf{V}\cdot\mathbf{n}=0 \text{ for } \nu=0) \quad \text{on } S \qquad (1.1)$$

Here t denotes time, $\mathbf{r}$ is the radius vector with respect to point O, $\mathbf{V}$ is the velocity in the system of coordinates $Ox_1x_2x_3$, P is the pressure, U is the potential of mass forces, $\mathbf{w}_0$ is the absolute acceleration of point O, ω is the absolute angular velocity of the body, $\omega^{\cdot}$ is its angular acceleration, S is the boundary of region D, $\mathbf{n}$ is the unit vector of the internal normal to S (Fig. 1). In the case of ideal fluid the no-slip condition is replaced by the condition of no flow.

We shall write the angular momentum $\mathbf{K}$ of the body with fluid with respect to the center of inertia O_1 of the entire system

$$\mathbf{K}=\mathbf{J}\cdot\boldsymbol{\omega}+\rho\int_D \mathbf{r}\times\mathbf{V}\,dv \qquad (\mathbf{J}=\mathbf{J}^{(1)}+\mathbf{J}^{(2)}) \qquad (1.2)$$

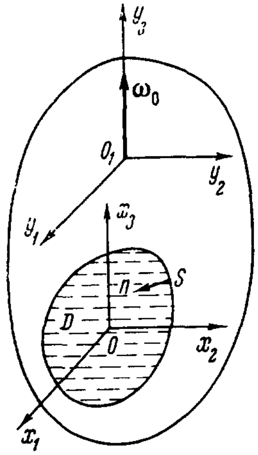

Fig. 1

Here $\mathbf{J}$ is the inertia tensor of the entire system with respect to point O_1, composed of the tensor of inertia of the body $\mathbf{J}^{(1)}$ and the fluid $\mathbf{J}^{(2)}$ with respect to the same point.

The second term in Eq. (1.2), called the hydrostatic moment, does not depend on the selection of the pole and may be computed with respect to point O. The equations of moments with respect to point O_1 is written in the system of coordinates $Ox_1x_2x_3$

$$\mathbf{K}^{\cdot} + \boldsymbol{\omega} \times \mathbf{K} = \mathbf{M} \qquad (1.3)$$

Here the dot denotes a derivative in the system $Ox_1x_2x_3$, $\mathbf{M}$ is the principal moment, with respect to point O_1 of all external forces acting on the body with the fluid.

Eqs. (1.1) to (1.3), together with the usual equations of motion of the center of inertia, kinematic relationships and initial conditions fully describe the dynamics of the body with the fluid.

Let the unperturbed motion of the body with fluid with respect to the center of inertia O_1 be a rotation of the whole system with constant angular velocity ω_0 around the axis O_1y_3 which passes through the point O_1 parallel to axis Ox_3. We shall examine the perturbed motion assuming that its deviations from the unperturbed motion are small and proportional to $e^{\lambda t}$ where λ is a complex number. Let us assume

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\Omega} e^{\lambda t}, \qquad \boldsymbol{\omega}^{\cdot} = \lambda \boldsymbol{\Omega} e^{\lambda t}, \qquad \boldsymbol{\omega}_0 = \omega_0 \mathbf{e}_3 \qquad (\omega_0 \geqslant 0)$$

$$\mathbf{V} = e^{\lambda t}\mathbf{v}, \qquad P = \rho\,[U - \mathbf{w}_0 \cdot \mathbf{r} + {}^1/_2\,(\boldsymbol{\omega} \times \mathbf{r})^2] + p e^{\lambda t} \qquad (1.4)$$

Here $\mathbf{e}_3$ is the unit vector of axis Ox_3, $\boldsymbol{\Omega}$ is a constant vector, $\mathbf{v}$ and p are functions of coordinates x_1, x_2, x_3, where all potential terms in Eq. (1.1) are taken with respect to pressure. Quantities $\boldsymbol{\Omega}$, $\mathbf{v}$ and p are considered small of the first order.

Substituting Eqs. (1.4) into Eqs. (1.1) to (1.3) and discarding small terms of higher order, we obtain the basic Eqs. in the form

$$\lambda \boldsymbol{\Omega} \times \mathbf{r} + 2\boldsymbol{\omega}_0 \times \mathbf{v} + \lambda \mathbf{v} = -\nabla p + \nu \Delta \mathbf{v}, \quad \operatorname{div} \mathbf{v} = 0 \quad \text{in } D$$

$$\mathbf{v} = 0 \quad (\mathbf{v} \cdot \mathbf{n} = 0 \ \text{for} \ \nu = 0) \quad \text{on } S \qquad (1.5)$$

$$\mathbf{K} = \mathbf{J} \cdot \boldsymbol{\omega}_0 + e^{\lambda t}(\mathbf{J} \cdot \boldsymbol{\Omega} + \mathbf{G}), \qquad \mathbf{G} = \rho \int_D \mathbf{r} \times \mathbf{v}\, dv$$

$$\boldsymbol{\omega}_0 \times (\mathbf{J} \cdot \boldsymbol{\omega}_0) + e^{\lambda t}\,[\lambda\,(\mathbf{J} \cdot \boldsymbol{\Omega} + \mathbf{G}) + \boldsymbol{\Omega} \times (\mathbf{J} \cdot \boldsymbol{\omega}_0) + \boldsymbol{\omega}_0 \times (\mathbf{J} \cdot \boldsymbol{\Omega} + \mathbf{G})] = \mathbf{M} \quad (1.6)$$

Without destroying generality we select as the unit of time the characteristic time of rotation of the body $T \sim 1/\omega_0$, as the unit of length the characteristic dimension of cavity l and as a unit of mass the mass of the entire system m. Then the ratio of the mass of fluid to the mass of the whole system has the order of magnitude $\rho l^3/m = \rho$, while the Reynolds number is equal to $l^2\nu^{-1}T^{-1} = \nu^{-1}$. The quantities ρ and ν may be considered nondimensional parameters.

2. On natural oscillations of the fluid. If the motion of the liquid is prescribed, the quantity λ and the vector $\boldsymbol{\Omega}$ are known. Determination of the motion of the liquid is reduced to a boundary value problem (1.5) for functions $\mathbf{v}$ and p. The solution of this problem is unique if and only if λ is not a characteristic value of the homogeneous problem

$$2\boldsymbol{\omega}_0 \times \mathbf{v} + \lambda \mathbf{v} = -\nabla p + \nu \Delta \mathbf{v}, \quad \operatorname{div} \mathbf{v} = 0 \quad \text{in } D$$

$$\mathbf{v} = 0 \quad (\mathbf{v} \cdot \mathbf{n} = 0 \ \text{for} \ \nu = 0) \quad \text{on } S \qquad (2.1)$$

Eqs. (2.1) describe natural oscillations of the fluid in a uniformly rotating vessel.

We shall prove that in the case of a viscous fluid all characteristic values of problem (2.1) are located in the half-strip $|\operatorname{Im} \lambda| \leqslant 2\omega_0$, $\operatorname{Re} \lambda \leqslant -C\nu$, and in case of the ideal fluid ($\nu = 0$) on the section $\operatorname{Re}\lambda = 0$, $|\lambda| \leqslant 2\omega_0$. Here $C > 0$ is a constant depending only on region D. The boundary S is assumed to be sufficiently smooth in this case. The formulated statement is known [1] and [6] for the case of ideal fluid.

For the purpose of proof the first Eq. of (2.1) is scalarly multiplied by $\mathbf{v}^*$ (the asterisk denotes complex-conjugate values everywhere) and the following Eqs., which are valid by virtue of the second Eq. of (2.1), are substituted into it

$$\mathbf{v}^*\nabla p = \operatorname{div}(p\mathbf{v}^*), \quad \mathbf{v}^*\Delta\mathbf{v} = \operatorname{div}(\mathbf{v}^* \times \operatorname{rot}\mathbf{v}) - |\operatorname{rot}\mathbf{v}|^2$$

Subsequently we integrate the equation obtained over the region D. Integrals of divergent terms become zero as a result of boundary condition (2.1) and we obtain

$$2\omega_0 \int_D \mathbf{e}_3\cdot(\mathbf{v}\times\mathbf{v}^*)\,dv + \lambda \int_D |\mathbf{v}|^2\,dv + \nu \int_D |\operatorname{rot}\mathbf{v}|^2\,dv = 0 \qquad (2.2)$$

Temporarily, let us designate by $\mathbf{a}$ and $\mathbf{b}$ the real and imaginary parts of vector $\mathbf{v} = \mathbf{a} + i\mathbf{b}$. Then we have

$$\mathbf{v}\times\mathbf{v}^* = (\mathbf{a}+i\mathbf{b})\times(\mathbf{a}-i\mathbf{b}) = -2i\,(\mathbf{a}\times\mathbf{b})$$

$$|\mathbf{e}_3\cdot(\mathbf{v}\times\mathbf{v}^*)| \leqslant 2\,|\mathbf{a}\times\mathbf{b}| \leqslant 2\,|\mathbf{a}|\cdot|\mathbf{b}| \leqslant |\mathbf{a}|^2 + |\mathbf{b}|^2 = |\mathbf{v}|^2 \qquad (2.3)$$

Separating in Eq. (2.2) the real and imaginary parts and using the first Eq. of (2.3) we obtain

$$\operatorname{Re}\lambda \int_D |\mathbf{v}|^2 dv = -\nu \int_D |\operatorname{rot}\mathbf{v}|^2 dv, \qquad \operatorname{Im}\lambda \int_D |\mathbf{v}|^2 dv = 4\omega_0 \int_D \mathbf{e}_3\cdot(\mathbf{a}\times\mathbf{b})\,dv \qquad (2.4)$$

Let λ be the eigenvalue of problem (2.1), $\mathbf{v}$ the eigenfunction corresponding to it (its norm is positive). Then, using inequality in (2.3), we obtain from the second Eq. of (2.4) the desired evaluation $|\operatorname{Im}\lambda| \leqslant 2\,\omega_0$, which is valid for both the ideal and the viscous fluid. In the case of ideal fluid ($\nu = 0$) it follows immediately from the first Eq. of (2.4) that $\operatorname{Re}\lambda = 0$. In the case of a viscous fluid we utilize the following inequality [9]:

$$\int_D |\operatorname{rot}\mathbf{v}|^2 dv \geqslant C \int_D |\mathbf{v}|^2 dv \qquad (2.5)$$

which is valid in case of sufficiently smooth boundary S of region D, when $\operatorname{div}\mathbf{v} = 0$ in D and the tangent to surface S of the component of vector $\mathbf{v}$ becomes zero on S. These conditions are satisfied in the case of a viscous fluid (see (2.1)). The constant $C > 0$ depends only on region D. In [9] the inequality (2.5) is proved for real vector functions, however it is apparent that it will also apply (with the same constant C) to complex vector functions. Applying inequality (2.5) to the first Eq. of (2.4) for $\mathbf{v} > 0$, we obtain the desired inequality $\operatorname{Re}\lambda \leqslant -C\nu$ for the viscous fluid. In this manner all natural oscillations of the rotating viscous liquid in the vessel decay not slower than $\exp(-C\nu t)$.

3. Ideal fluid. Let λ be fixed and lie outside the section $\operatorname{Re}\lambda = 0$, $|\lambda| \leqslant 2\omega_0$, where all the characteristic values of problem (2.1) are located. Then neither λ, nor $(-\lambda)$ are characteristic values of problem (2.1). We shall show that the solution of the hydrodynamic problem and the vector $\mathbf{G}$ entering into equations of motion of the body (1.6) can be expressed through some universal functions. In Section 3 this problem is solved for an ideal fluid (superscript $^\circ$ at $\mathbf{v}$, p, $\mathbf{G}$ refers to the ideal fluid), in Section 4 it is solved for a fluid with low viscosity.

In case of ideal fluid, Eqs. (1.5) have the form

$$2\omega_0\times\mathbf{v}^\circ + \lambda\mathbf{v}^\circ + \mathbf{g} = 0, \qquad \operatorname{div}\mathbf{v}^\circ = 0 \quad \text{in } D$$

$$\mathbf{v}^\circ\cdot\mathbf{n} = 0 \quad \text{on } S, \qquad \mathbf{g} \equiv \lambda\Omega\times\mathbf{r} + \nabla p^\circ \qquad (3.1)$$

The first Eq. of (3.1) is solved with respect to $\mathbf{v}^\circ$. Scalar and vector pre-multiplication by by ω_0 gives

$$\lambda\omega_0\cdot\mathbf{v}^\circ + \omega_0\cdot\mathbf{g} = 0, \quad 2\omega_0(\omega_0\cdot\mathbf{v}^\circ) - 2\omega_0^2\mathbf{v}^\circ + \lambda\omega_0\times\mathbf{v}^\circ + \omega_0\times\mathbf{g} = 0 \quad (3.2)$$

Into the second Eq. of (3.2) the product $\omega_0\cdot\mathbf{v}^\circ$ is substituted from the first Eq. of (3.2) and $\omega_0\times\mathbf{v}^\circ$ from the first Eq. of (3.1) and then the second Eq. of (3.2) is solved with respect to $\mathbf{v}^\circ$

$$\mathbf{v}^\circ = [2\omega_0\times\mathbf{g} - \lambda\mathbf{g} - 4\lambda^{-1}\omega_0(\omega_0\cdot\mathbf{g})]\,/\,(\lambda^2 + 4\omega_0^2) \qquad (3.3)$$

Let us introduce a linear transformation $\mathbf{L}$ and a complex number σ

$$\sigma = 2\omega_0/\lambda, \qquad \mathbf{L}\cdot\mathbf{a} = \mathbf{a} + \sigma^2\mathbf{e}_3(\mathbf{e}_3\cdot\mathbf{a}) + \sigma(\mathbf{a}\times\mathbf{e}_3)$$

$$\mathbf{L} = \begin{pmatrix} 1 & \sigma & 0 \\ -\sigma & 1 & 0 \\ 0 & 0 & 1+\sigma^2 \end{pmatrix} \qquad (3.4)$$

Here $\mathbf{a}$ is an arbitrary vector, the matrix $\mathbf{L}$ determines the transformation in the system of coordinates $Ox_1x_2x_3$. Matrix $\mathbf{L}$, dependent on σ, has the properties

$$\mathbf{L}'(\sigma) = \mathbf{L}(-\sigma), \qquad \mathbf{a}\cdot(\mathbf{L}\cdot\mathbf{b}) = (\mathbf{L}'\cdot\mathbf{a})\cdot\mathbf{b} \tag{3.5}$$

Here the prime denotes the transposed matrix, $\mathbf{a}$ and $\mathbf{b}$ are arbitrary vectors. In the notation of (3.4), Eq. (3.3), the equation of continuity and boundary condition (3.1) take on the form

$$\mathbf{v}^\circ = -\lambda^{-1}(1+\sigma^2)^{-1}\mathbf{L}\cdot\mathbf{g}, \qquad \mathbf{g} = \lambda\boldsymbol{\Omega}\times\mathbf{r} + \nabla p^\circ$$
$$\operatorname{div}(\mathbf{L}\cdot\mathbf{g}) = 0 \quad \text{in } D, \qquad \mathbf{n}\cdot(\mathbf{L}\cdot\mathbf{g}) = 0 \quad \text{on } S \tag{3.6}$$

Solution of Eqs. (3.6) can be presented in the form

$$p^\circ = -\lambda\sum_{j=1}^{3}\Omega_j\varphi_j, \qquad \mathbf{v}^\circ = \frac{1}{1+\sigma^2}\sum_{j=1}^{3}\Omega_j\mathbf{L}\cdot(\nabla\varphi_j - \mathbf{e}_j\times\mathbf{r}) \tag{3.7}$$

Here $\mathbf{e}_j$ are unit vectors of axes Ox_j, $\Omega_j = \boldsymbol{\Omega}\cdot\mathbf{e}_j$ are projections of vector $\boldsymbol{\Omega}$ on these axes, while functions φ_j satisfy linear boundary value problems (3.8)

$$\operatorname{div}[\mathbf{L}\cdot(\nabla\varphi_j - \mathbf{e}_j\times\mathbf{r})] = 0 \quad \text{in } D, \quad \mathbf{n}\cdot[\mathbf{L}\cdot(\nabla\varphi_j - \mathbf{e}_j\times\mathbf{r})] = 0 \quad \text{on } S \quad (j=1,2,3)$$

Substituting Expression (3.4) for $\mathbf{L}$, Eq. (3.8) can be written in the form

$$\Delta\varphi_j + \sigma^2(\partial^2\varphi_j/\partial x_3^2) - 2\sigma\delta_{j3} = 0 \quad \text{in } D \qquad (j=1,2,3) \tag{3.9}$$

Here Δ is the Laplace operator with respect to variables x_1, x_2, x_3 and δ_{j3} is the Kronecker symbol. Substituting $\mathbf{v}^\circ$ from (3.7) into Eq. (1.6) we obtain for $\mathbf{G}$

$$\mathbf{G}^\circ = \rho\mathbf{I}^\circ\cdot\boldsymbol{\Omega} = \rho\sum_{j,k=1}^{3}\mathbf{e}_j I_{jk}^\circ\Omega_k, \qquad I_{jk}^\circ = \frac{1}{1+\sigma^2}\int_D(\mathbf{e}_j\times\mathbf{r})\cdot[\mathbf{L}\cdot(\nabla\varphi_k - \mathbf{e}_k\times\mathbf{r})]\,dv$$
$$(j,k=1,2,3) \tag{3.10}$$

Here $\mathbf{I}^\circ$ is a tensor, I_{jk}° are its components in the system of coordinates $Ox_1x_2x_3$. In this manner the determination of $\mathbf{v}^\circ$, p° and $\mathbf{G}^\circ$ is reduced to the solution of boundary value problems (3.8) and to computation on integrals (3.10). Relationships (3.10) can be given a different form if functions φ_j', are introduced into the investigation and these functions satisfy the boundary conditions (3.11)

$$\operatorname{div}[\mathbf{L}'\cdot(\nabla\varphi_j' - \mathbf{e}_j\times\mathbf{r})] = 0 \quad \text{in } D, \quad \mathbf{n}\cdot[\mathbf{L}'\cdot(\nabla\varphi_j' - \mathbf{e}_j\times\mathbf{r})] = 0 \quad \text{on } S \quad (j=1,2,3)$$

From Eqs. (3.5), (3.8) and (3.11) it is apparent that functions φ_j' for some value of σ are simultaneously functions of φ_j for the value $(-\sigma)$. From these same equations, identities follow which are valid for any function f

$$\nabla f\cdot[\mathbf{L}'\cdot(\mathbf{e}_j\times\mathbf{r})] = \nabla f\cdot[\mathbf{L}'\cdot(\nabla\varphi_j')] - \operatorname{div}[f\mathbf{L}'\cdot(\nabla\varphi_j' - \mathbf{e}_j\times\mathbf{r})]$$

$$\int_D\nabla f\cdot[\mathbf{L}'\cdot(\mathbf{e}_j\times\mathbf{r})]\,dv = \int_D\nabla f\cdot[\mathbf{L}'\cdot(\nabla\varphi_j')]\,dv \tag{3.12}$$

Application of the second identity of (3.5) and the second identity of (3.12) to integrals (3.10) for $f = \varphi_k$, we obtain

$$I_{jk}^\circ = \frac{1}{1+\sigma^2}\int_D\{\nabla\varphi_k\cdot[\mathbf{L}'\cdot(\nabla\varphi_j')] - (\mathbf{e}_k\times\mathbf{r})\cdot[\mathbf{L}'\cdot(\mathbf{e}_j\times\mathbf{r})]\}\,dv \quad (j,k=1,2,3) \tag{3.13}$$

Functions φ_j, φ_j' and tensor $\mathbf{I}^\circ$ depend only on the shape of the cavity and the value of σ. From Eqs. (3.5) and (3.13) it follows that

$$I_{jk}^\circ(-\sigma) = I_{kj}^\circ(\sigma) \qquad (j,k=1,2,3) \tag{3.14}$$

Let functions φ_j and tensor $\mathbf{I}^\circ$ correspond to some σ. Applying to Eqs. (3.8) the operation of complex conjugation we obtain that functions φ_j^* correspond to the value σ^*

$$I_{jk}^\circ(\sigma^*) = [I_{jk}^\circ(\sigma)]^* \qquad (j,k=1,2,3) \tag{3.15}$$

If σ is real ($\sigma^* = \sigma$) then it follows from what was said before that all functions φ_j and components of the tensor $\mathbf{I}^\circ$ are real in this case. If on the other hand σ is a purely imaginary number ($\sigma^* = -\sigma$), then by comparing Eqs. (3.14) and (3.15) we obtain

$$[I_{jk}^\circ(\sigma)]^* = I_{kj}^\circ(\sigma) \quad \text{for } \operatorname{Re}\sigma = 0 \quad (j,k=1,2,3) \tag{3.16}$$

In case of irrotational motion ($\omega_0 = \sigma = 0$) operators $\mathbf{L}$ and $\mathbf{L}'$ from (3.4) and (3.5) turn into unit operators. In this case problems (3.8) and (3.11) for functions φ_j and φ_j' transform into problems of Neumann for harmonic functions which are called Zhukovskii's potentials for a given cavity [3]. It follows then from Eq. (3.13) that for $\sigma = 0$ with accuracy to a factor of ρ the tensor $\mathbf{I}^\circ$ is equal to the difference between the tensor of apparent additional masses and the tensor of inertia for the given cavity (with respect to point 0).

4. Viscous fluid. Solution of problem (1.5) for the case of a fluid of low viscosity ($\nu \ll 1$) is sought by the boundary layer method which has repeatedly been applied in similar problems ([3 to 6 and 8]). We assume (the superscript indicates the order of approximation)

$$\mathbf{v} = (\mathbf{v}^\circ + \nu^{1/2}\mathbf{v}^1 + \ldots) + \mathbf{w}, \qquad p = (p^\circ + \nu^{1/2} p^1 + \ldots) + q \tag{4.1}$$

We shall require that sums within parenthesis in Eqs. (4.1) satisfy (without terms $\mathbf{w}$ and q) the Navier-Stokes equations. Substituting these sums into Eqs. (1.5) and equating terms with equal powers of ν we obtain for $\mathbf{v}^\circ$ and p° Eqs. (3.1) and for $\mathbf{v}^1$ and p^1 Eqs.

$$2\boldsymbol{\omega}_0 \times \mathbf{v}^1 + \lambda\mathbf{v}^1 = -\nabla p^1, \qquad \operatorname{div}\mathbf{v}^1 = 0 \tag{4.2}$$

Analogously we may write equations also for the following terms of expansions (4.1). Considering terms $\mathbf{w}$ and q, for functions $\mathbf{v}$ and p to satisfy Navier-Stokes equations, it is necessary to require that

$$2\boldsymbol{\omega}_0 \times \mathbf{w} + \lambda\mathbf{w} = -\nabla q + \nu\Delta\mathbf{w}, \qquad \operatorname{div}\mathbf{w} = 0 \tag{4.3}$$

The functions $\mathbf{w}$ and q are also considered as expanded in powers of $\nu^{1/2}$

$$\mathbf{w} = \mathbf{w}^\circ + \nu^{1/2}\mathbf{w}^1 + \cdots, \qquad q = q^\circ + \nu^{1/2}q^1 + \cdots \tag{4.4}$$

where the coefficients of expansions are functions of the boundary layer type which rapidly tend to zero outside the boundary layer region D_S. The region D_S adjoins the walls S and has a thickness of the order $\nu^{1/2}$.

The boundary conditions for coefficients of expansion (4.1) and (4.4) are obtained by the following recursion method. The functions $\mathbf{v}^\circ$ and p° are subjected to the condition $\mathbf{v}^\circ \cdot \mathbf{n} = 0$ on S, then they coincide with solution (3.7) for the ideal fluid. The condition $(\mathbf{w}^\circ)_\tau = -\mathbf{v}^\circ$ on S is placed on functions $\mathbf{w}^\circ$ and q° and also $\mathbf{w}^\circ \to 0$, $q^\circ \to 0$ outside the region D_S. Index τ denotes projection of the vector on the plane which is tangent to the surface S. Functions $\mathbf{v}^1$ and p^1 are subjected to the condition $\nu^{1/2}\mathbf{v}^1 \cdot \mathbf{n} = -\mathbf{w}^\circ \cdot \mathbf{n}$ on S. In general, in the k-th approximation functions $\mathbf{v}^k$ and p^k must compensate for the discrepancy in satisfying the boundary condition $\mathbf{v} \cdot \mathbf{n} = 0$ on S, which was caused by preceding terms of expansions $\mathbf{v}^i$ and $\mathbf{w}^i$ ($i = 0, 1, \ldots, k-1$). Functions $\mathbf{w}^k$ and q^k must satisfy conditions $\mathbf{w}^k \to 0$ and $q^k \to 0$ outside of the boundary layer D_S and compensate for the discrepancy in satisfying the condition $\mathbf{v}_\tau = 0$, which arises due to functions $\mathbf{v}^k$, $\mathbf{v}^i$ and $\mathbf{w}^i$ for $i = 0, 1, \ldots, k-1$. Without examining questions regarding the mathematical basis of expansions (4.1) and (4.4), we note that the boundary layer method was applied to hydrodynamic problems leading as a rule to physically correct results. For one problem of fluid motion in a cavity of a rotating solid body the agreement of results from calculations by the boundary layer method with experimental data is noted in [5]. In the following, in addition to $\mathbf{v}^\circ$ and p° the terms $\mathbf{v}^1$, p^1 and $\mathbf{w}^\circ$ and q° will also be taken into account. These terms will be denoted simply by $\mathbf{w}$ and q. In the region of boundary layer D_S we shall introduce curvilinear orthogonal coordinates ξ, η, ζ in such a manner that $\zeta = 0$ on the surface of walls S. In this case $\zeta > 0$ in the region D_S. Let w_ξ, w_η and ω_ζ be the components of vector $\mathbf{w}$ in these coordinates, and H_ξ, H_η and H_ζ the corresponding coeeficients of Lamé, H_ξ°, H_η° and H_ζ° are the values of these coefficients for $\zeta = 0$.

Without destroying generality it is assumed that $H_\zeta^\circ = 1$, then ζ is the distance along the normal $\mathbf{n}$ from the surface S.

In Eqs. (4.3) we pass to coordinates ξ, η, ζ and then make a substitution of variables

$$\zeta = \nu^{1/2}\alpha, \qquad w_\zeta = \nu^{1/2}w_\alpha \tag{4.5}$$

and in the equations pass to the limit for $\nu \to 0$. We obtain

$$-2\omega_0\mathbf{n}\cdot\mathbf{e}_3 w_\eta + \lambda w_\xi = -\frac{\partial q}{\partial \xi} + \frac{\partial^2 w_\xi}{\partial \alpha^2}, \qquad 2\omega_0\mathbf{n}\cdot\mathbf{e}_3 w_\xi + \lambda w_\eta = -\frac{\partial q}{\partial \eta} + \frac{\partial^2 w_\eta}{\partial \alpha^2}$$

$$\partial q / \partial\alpha = 0, \qquad \operatorname{Div}\mathbf{w}_\tau + \partial w_\alpha / \partial\alpha = 0 \qquad \mathbf{w}_\tau = (w_\xi, w_\eta)$$

$$\operatorname{Div}\mathbf{w}_\tau = \frac{1}{H_\xi^\circ H_\eta^\circ}\left[\frac{\partial (H_\eta^\circ w_\xi)}{\partial \xi} + \frac{\partial (H_\xi^\circ w_\eta)}{\partial \eta}\right] \tag{4.6}$$

$$w_\xi = -v_\xi^\circ, \quad w_\eta = -v_\eta^\circ \quad \text{for} \quad \alpha = 0; \qquad w_\xi, w_\eta, w_\alpha, q \to 0 \quad \text{for} \quad \alpha \to \infty$$

Here the boundary conditions for $\mathbf{w}$ and q indicated above are also written out. The two-dimensional divergence operation with respect to variables ξ and η is denoted through Div for two-dimensional vector fields on the surface S, $\mathbf{w}_\tau$ is a vector with components w_ξ, w_η and v_ξ°, v_η° are corresponding components of vector $\mathbf{v}^\circ$ where $v_\zeta^\circ = 0$ on S by virtue of condition $\mathbf{v}^\circ \cdot \mathbf{n} = 0$. From the third Eq. of (4.6) it follows that q is independent of α. Taking into account the boundary condition for $\alpha = \infty$ we obtain $q \equiv 0$. Then the first two Eqs. of (4.6) transform into a system of two ordinary linear differential equations of second order with constant coefficients. In these equations the role of the argument is played by α while ξ and η on which $\mathbf{n}$ is dependent, enter as parameters. The solution of this fourth order equation with two boundary conditions (4.6) for $\alpha = 0$ and with two conditions for $\alpha = \infty$ has the form

$$w_\xi = -{}^1/_2 (v_\xi^\circ + iv_\eta^\circ) E_1 - {}^1/_2 (v_\xi^\circ - iv_\eta^\circ) E_2$$

$$w_\eta = {}^1/_2 (iv_\xi^\circ - v_\eta^\circ) E_1 - {}^1/_2 (iv_\xi^\circ + v_\eta^\circ) E_2$$

$$E_k = \exp(\mu_k \alpha) = \exp(\mu_k \zeta \nu^{-1/2}) \qquad (k = 1, 2) \tag{4.7}$$

$$\mu_{1,2} = \sqrt{\lambda \pm 2i\omega_0 \mathbf{n}\cdot\mathbf{e}_3} = \sqrt{2\omega_0}\sqrt{\sigma^{-1} \pm i\mathbf{n}\cdot\mathbf{e}_3}$$

As μ_1 and μ_2 those branches of the root are taken for which $\operatorname{Re}\mu_k \leqslant 0$ for $k = 1, 2$. Values μ_1 and μ_2 depend on the point of the surface S.

The solution (4.7) can be written in vector form

$$\mathbf{w}_\tau = -{}^1/_2 \mathbf{v}^\circ \ (E_1 + E_2) - {}^1/_2 i \ (\mathbf{v}^\circ \times \mathbf{n}) \ (E_1 - E_2) \tag{4.8}$$

Taking into account the notation in (4.7), we substitute solution (4.8) into the fourth Eq. of (4.6) and integrate it with respect to α with the boundary condition $\omega_\alpha = 0$ for $\alpha = \infty$. Then we obtain function w_α and from equation (4.5) also w_ζ in the form

$$w_\zeta = {}^1/_2 \nu^{1/2} \operatorname{Div} [(\mu_1^{-1}E_1 + \mu_2^{-1}E_2)\, \mathbf{v}^\circ + i\, (\mu_1^{-1}E_1 - \mu_2^{-1}E_2)\, \mathbf{v}^\circ \times \mathbf{n}] \tag{4.9}$$

Eqs. $q = 0$ and (4.7) to (4.9) completely determine the functions $\mathbf{w}$ and q. If $\operatorname{Re}\mu_k < 0$ for $k = 1, 2$, then the functions $\mathbf{w}_\tau$ and w_ζ decrease rapidly (exponentially) with increasing ζ. Outside the boundary layer D_S, i.e. for $\zeta \gg \nu^{1/2}$ it is permissible to set $\mathbf{w} \equiv 0$ with an error smaller than any power of ν. If however $\operatorname{Re}\mu_1 = 0$ or $\operatorname{Re}\mu_2 = 0$, then the function $\mathbf{w}$ does not decrease with increasing distance from the walls of the cavity, which destroys the initial assumption of a boundary layer. This will occur, as follows from Eqs. (4.7), under conditions

$$\operatorname{Re}\lambda \leqslant 0, \qquad \operatorname{Im}\lambda = \pm 2\omega_0 \mathbf{n}\cdot\mathbf{e}_3 \tag{4.10}$$

If λ does not lie within the half-strip $\operatorname{Re}\lambda \leqslant 0$, $|\operatorname{Im}\lambda| \leqslant 2\omega_0$, for which it is known that it contains all eigenvalues of the problem (see Section 2), then conditions (4.10) are not satisfied for any $\mathbf{n}$ and the boundary layer will not have singularities anywhere. If however λ is located in the half-strip indicated, then in the vicinity of some points of the walls in which the normal $\mathbf{n}$ satisfies the second condition of (4.10), the boundary layer will have a singularity (its thickness will tend to infinity). In case of smooth surfaces these critical points form usually closed curves (for example, for a spherical cavity there are two circumferences with centers on the axis of revolution). A more detailed analysis of the solution near these points shows [4 and 5] that the solution here remains bounded but has a more complex character than in the boundary layer. In the following we neglect the influence of these singularities.

Functions $\mathbf{v}^1$ and p^1 satisfy Eq. (4.2) and the boundary condition stated above, which, taking into consideration Eqs. (4.9) and (4.7), for $\zeta = 0$ can be written in the form

$$\mathbf{v}^1 \cdot \mathbf{n} = -\nu^{-1/2}\mathbf{w}\cdot\mathbf{n} = -\nu^{-1/2} w_\zeta = -\operatorname{Div}\mathbf{A} \quad \text{on } S$$

$$\mathbf{A} = {}^1/_2 [(\mu_1^{-1} + \mu_2^{-1})\, \mathbf{v}^\circ + i\, (\mu_1^{-1} - \mu_2^{-1})\, \mathbf{v}^\circ \times \mathbf{n}] \tag{4.11}$$

The first Eq. of (4.2) is solved with respect to $\mathbf{v}^1$ and $\mathbf{v}^1$ is substituted into the second Eq. of (4.2). In the notation of (3.4) we obtain in analogy to (3.6):

$$\mathbf{v}^1 = -\lambda^{-1}(1 + \sigma^2)^{-1}\mathbf{L}\cdot(\nabla p^1), \qquad \operatorname{div}[\mathbf{L}\cdot(\nabla p^1)] = 0 \tag{4.12}$$

Without solving the problem (4.11), and (4.12) for $\mathbf{v}^1$ and p^1 we can find the vector $\mathbf{G}$ from (1.6) with an error of order $(\nu^{1/2})$. Since in the region D_S with a volume $0(\nu^{1/2})$ we have $|\mathbf{w}_\tau| \sim 1$ and $w_\zeta \sim \nu^{1/2}$, while outside of this region we can assume $\mathbf{w} = 0$, then to obtain the desired accuracy we assume

$$\mathbf{v} = \mathbf{v}^\circ + \mathbf{w}_\tau + \nu^{1/2}\mathbf{v}^1$$

$$\mathbf{G} = \mathbf{G}^\circ + \rho \int_{D_s} \mathbf{r} \times \mathbf{w}_\tau \, dv + \rho\nu^{1/2} \int_D \mathbf{r} \times \mathbf{v}^1 \, dv \tag{4.13}$$

The term $\mathbf{G}^\circ$ corresponds to vector $\mathbf{v}^\circ$ and is determined by Eq. (3.10). Without loss in accuracy integration with respect to D_s in (4.13) can be replaced by integration with respect to ζ from O to ∞ over the surface S. Substituting Expression (4.8) for $\mathbf{w}\tau$ we obtain after integration with respect to ζ, taking into account expression (4.11) for $\mathbf{A}$

$$\int_{D_s} \mathbf{r} \times \mathbf{w}_\tau \, dv = \oint_S \mathbf{r} \times \left(\int_0^\infty \mathbf{w}_\tau \, d\zeta \right) ds = \nu^{1/2} \oint_S \mathbf{r} \times \mathbf{A} \, ds \tag{4.14}$$

In the third term of Eq. (4.13) for $\mathbf{G}$ we substitute $\mathbf{v}^1$ from (4.12) and expand the obtained vector with respect to unit vectors $\mathbf{e}_j$

$$\int_D \mathbf{r} \times \mathbf{v}^1 \, dv = \frac{-1}{\lambda(1+\sigma^2)} \sum_{j=1}^3 \mathbf{e}_j \int_D (\mathbf{e}_j \times \mathbf{r}) \cdot [\mathbf{L} \cdot (\nabla p^1)] \, dv$$

Utilizing the second identity of (3.5) and subsequently the second identity of (3.12) for $f = p^1$, we obtain

$$\int_D \mathbf{r} \times \mathbf{v}^1 dv = -\frac{1}{\lambda(1+\sigma^2)} \sum_{j=1}^3 \mathbf{e}_j \int_D \nabla p^1 \cdot [\mathbf{L}' \cdot (\nabla \varphi_j')] \, dv \tag{4.15}$$

The expression under the integral in (4.15) is transformed with the aid of Eqs. (3.5) and (4.12)

$$\nabla p^1 \cdot [\mathbf{L}' \cdot (\nabla \varphi_j')] = [\mathbf{L} \cdot (\nabla p^1)] \cdot \nabla \varphi_j' = \operatorname{div} [\varphi_j' \mathbf{L} \cdot (\nabla p^1)] = -\lambda (1+\sigma^2) \operatorname{div} (\varphi_j' \mathbf{v}^1)$$

The obtained relationship is substituted into Eq. (4.15) and the theorem of Gauss-Ostrogradskii and boundary condition (4.11) are applied

$$\int_D \mathbf{r} \times \mathbf{v}^1 dv = \sum_{j=1}^3 \mathbf{e}_j \oint_S \varphi_j' \operatorname{Div} \mathbf{A} ds \tag{4.16}$$

We note the following identity which applies to any vector-function $\mathbf{a}$ and scalar function f

$$f \operatorname{Div} \mathbf{a} = \operatorname{Div} (f\mathbf{a}) - \mathbf{a} \cdot \operatorname{Grad} f = \operatorname{Div} (f\mathbf{a}) - \mathbf{a} \cdot (\nabla f - \mathbf{n}\, \partial f / \partial n) \tag{4.17}$$

Here Grad is the operator for taking the gradient along the surface S [10]. For any vector field $\mathbf{b}$ given on a closed surface S, the following identity is valid

$$\oint_S \operatorname{Div} \mathbf{b} \, ds = 0 \tag{4.18}$$

which follows from the theorem of Gauss-Ostrogradskii for vector fields on the surface [10].

Identity (4.17) for $\mathbf{a} = \mathbf{A}$ and $f = \varphi_j'$ and identity (4.18) are applied for transformation of integral (4.16). We also note that from boundary condition (3.1) and Eq. (4.11) for $\mathbf{A}$ it follows that $\mathbf{A} \cdot \mathbf{n} = 0$ on S. Then integral (4.16) takes the form

$$\int_D \mathbf{r} \times \mathbf{v}^1 \, dv = -\sum_{j=1}^3 \mathbf{e}_j \oint_S \nabla \varphi_j' \cdot \mathbf{A} \, ds \tag{4.19}$$

Substituting Expressions (4.14) and (4.19) into Eq. (4.13) and decomposing the vector $\mathbf{r} \times \mathbf{A}$ with respect to coordinate axes we obtain with accuracy to small terms of higher order:

$$\mathbf{G} = \mathbf{G}^\circ - \rho\nu^{1/2} \sum_{j=1}^3 \mathbf{e}_j \oint_S (\nabla \varphi_j' - \mathbf{e}_j \times \mathbf{r}) \cdot \mathbf{A} \, ds \tag{4.20}$$

Into Eq. (4.20) we substitute vectors $\mathbf{A}$ from (4.11) and $\mathbf{v}^\circ$ from (3.7). Finally we obtain

$$\mathbf{G} = \mathbf{G}^\circ + \mathbf{G}^1, \qquad \mathbf{G}^1 = \rho\nu^{1/2} \mathbf{I}^1 \cdot \boldsymbol{\Omega} = \rho\nu^{1/2} \sum_{j,k=1}^3 \mathbf{e}_j I_{jk}^1 \Omega_k$$

$$I_{jk}{}^1 = -\frac{1}{2(1+\sigma^2)}\oint_S (\nabla\varphi_j' - \mathbf{e}_j\times\mathbf{r})\cdot\{(\mu_1^{-1}+\mu_2^{-1})\,\mathbf{L}\cdot(\nabla\varphi_k -$$

$$-\mathbf{e}_k\times\mathbf{r}) + i(\mu_1^{-1}-\mu_2^{-1})\,[\mathbf{L}\cdot(\nabla\varphi_k - \mathbf{e}_k\times\mathbf{r})]\times\mathbf{n}\}\,ds \quad (j,k=1,2,3) \tag{4.21}$$

Here $\mathbf{I}^1$ is a tensor, $I_{jk}{}^1$ are its components in the system of coordinates $Ox_1x_2x_3$, functions μ_1 and μ_2 are determined by Eqs. (4.7). The values of $I_{jk}{}^1$ and also of $I_{jk}{}^\circ$ from (3.10) and (3.13) depend only on the shape of the cavity and the value of σ. Since the vector $\mathbf{G}$ is independent of the selection of point O, tensors $\mathbf{I}^\circ$ and $\mathbf{I}^1$ also have the same property. It is sufficient for their determination to find functions φ_j and φ_j', which satisfy the boundary value problems (3.8) and (3.11) and are related to the motion of an ideal fluid. From Eqs. (3.10) and (4.21) it follows that

$$\mathbf{G} = \rho\mathbf{I}\cdot\Omega, \qquad \mathbf{I} = \mathbf{I}^\circ + \nu^{1/}\sigma \tag{4.22}$$

The velocity and pressure of the fluid are determined by Eqs. (4.1), where functions $\mathbf{v}^\circ$ and p° are given by Eqs. (3.7), $\mathbf{w}$ is determined by Eqs. (4.7) to (4.9), $q\equiv 0$, and $\mathbf{v}^1$ and p^1 satisfy the boundary value problem (4.11), (4.12). This problem, just like problems (3.8) and (3.11) for functions φ_j and φ_j' has a unique solution if λ lies outside the section $\mathrm{Re}\,\lambda = 0$, $|\lambda| \leqslant 2\omega_0$. In this case σ, according to Eq. (3.4), lies outside the set $\mathrm{Re}\,\sigma = 0$, $|\sigma| \geqslant 1$.

Let functions φ_j and φ_j' be known for the perscribed shape of the cavity (in a unique or even nonunique manner). Then from Eqs. (3.10), (3.13), (4.21) and (4.22) we may compute the tensors $\mathbf{I}^\circ$, $\mathbf{I}^1$ and the vector $\mathbf{G}$, and then by means of Eqs. (1.6) we may examine various dynamic problems of a body with a fluid.

We note that tensors $\mathbf{I}^\circ$ and $\mathbf{I}^1$ as functions of a complex parameter σ can have singularities only for $\mathrm{Re}\,\sigma = 0$, $|\sigma| \geqslant 1$ and also for $\sigma = 0$, i.e. for $\lambda = \infty$.

If constant rotation is absent ($\omega_0 = \sigma = 0$), then solutions of Section 4, as can be readily verified, transform into corresponding results of [8] where oscillations of a nonrotating body with fluid were examined.

5. **Particular shapes of cavities.** As examples we shall examine functions φ_j and tensors $\mathbf{I}^\circ$ and $\mathbf{I}^1$ for ellipsoidal and spherical cavities.

1. Let x_3 be the axis of symmetry of the cavity, i.e. it follows from the fact that if point (x_1, x_2, x_3) lies on the surface S that point $(-x_1, -x_2, -x_3)$ also lies on S. It is not difficult to see that in this case the solutions of problems (3.8) have the following properties in D:

$$\varphi_j(-x_1, -x_2, x_3) = -\varphi_j(x_1, x_2, x_3), \qquad \varphi_3(-x_1, -x_2, x_3) = \varphi_3(x_1, x_2, x_3)$$
$$(j = 1,2) \tag{5.1}$$

Functions φ_j' have the same properties. It follows from (3.10), (4.21) and (5.1) that

$$I_{3j}{}^\circ = I_{j3}{}^\circ = I_{3j}{}^1 = I_{j3}{}^1 = 0 \qquad (j = 1,2) \tag{5.2}$$

i.e. the axis x_3 is the principal axis of tensors $\mathbf{I}^\circ$ and $\mathbf{I}^1$.

2. Let the walls of the cavity form an ellipsoid

$$x_1^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2 = 1 \tag{5.3}$$

Functions φ_j are sought in the form (analogous solutions for an ellipsoidal cavity are known [11 and 3])

$$\varphi_1 = (b_{11}x_1 + b_{12}x_2)\,x_3, \quad \varphi_2 = (b_{21}x_1 + b_{22}x_2)\,x_3, \quad \varphi_3 = b_{31}x_1^2 + b_{32}x_2^2 + b_{33}x_1x_2 \tag{5.4}$$

In this case Eqs. (3.9) are satisfied for $j = 1, 2$, while for $j = 3$ we obtain from Eq. (3.9)

$$b_{31} + b_{32} = \sigma \tag{5.5}$$

Into boundary conditions (3.8) let us substitute the components of normal $\mathbf{n}$ to the surface of the ellipsoid (5.3), Expression (3.4) for $\mathbf{L}$ and functions (5.4). Then these conditions transform into homogeneous polynomials of second degree with respect to coordinates x_i. Equating to zero coefficients of these polynomials we obtain after simple transformations

$$\begin{gathered}
\frac{b_{11}+\sigma b_{12}+\sigma}{a_1^2} + \frac{(1+\sigma^2)\,b_{11}}{a_3^2} = \frac{b_{12}+1-\sigma b_{11}}{a_2^2} + \frac{(1+\sigma^2)(b_{12}-1)}{a_3^2} = 0 \\
\frac{b_{21}-1+\sigma b_{22}}{a_1^2} + \frac{(1+\sigma^2)(b_{21}+1)}{a_3^2} = \frac{b_{22}-\sigma b_{21}+\sigma}{a_2^2} + \frac{(1+\sigma^2)\,b_{22}}{a_3^2} = 0 \\
2b_{31}+\sigma b_{33}-\sigma = 0, \qquad 2b_{32}-\sigma b_{33}-\sigma = 0 \\
(b_{33}+1+2\sigma b_{32})/a_1^2 + (b_{33}-1-2\sigma b_{31})/a_2^2 = 0
\end{gathered} \tag{5.6}$$

Solving the first two linear Eqs. of (5.6) with respect to b_{11} and b_{12}, the second two with respect to b_{21} and b_{22} and the last three with respect to b_{31}, b_{32} and b_{33} we find

$$b_{11} = -\frac{2\sigma}{a_1^2 a_3^2 N}, \qquad b_{12} = \frac{1}{N}\left(\frac{1}{a_1^2 a_3^2} - \frac{1}{a_1^2 a_2^2} - \frac{1}{a_2^2 a_3^2} + \frac{1+\sigma^2}{a_3^4}\right)$$

$$b_{21} = \frac{1}{N}\left(\frac{1}{a_1^2 a_2^2} + \frac{1}{a_1^2 a_3^2} - \frac{1}{a_2^2 a_3^2} - \frac{1+\sigma^2}{a_3^4}\right), \qquad b_{22} = -\frac{2\sigma}{a_2^2 a_3^2 N}$$

$$b_{31} = \frac{\sigma a_2^2}{a_1^2 + a_2^2}, \qquad b_{32} = \frac{\sigma a_1^2}{a_1^2 + a_2^2}, \qquad b_{33} = \frac{a_1^2 - a_2^2}{a_1^2 + a_2^2} \tag{5.7}$$

$$N = \frac{1}{a_1^2 a_2^2} + \frac{1}{a_1^2 a_3^2} + \frac{1}{a_2^2 a_3^2} + \frac{1+\sigma^2}{a_3^4} = \frac{(a_1^2 + a_3^2)(a_2^2 + a_3^2)}{a_1^2 a_2^2 a_3^4} + \frac{\sigma^2}{a_3^4}$$

Coefficients (5.7) also satisfy condition (5.5). Eqs. (5.4) and (5.7) determine functions φ_j for ellipsoidal cavity. Functions φ_j' are obtained from Eqs. (5.4), if in Expressions (5.7) σ is everywhere replaced by $-\sigma$. The solutions found exist for all σ for which $N \neq 0$.

Substituting Expression (3.4) for $\mathbf{L}$ and (5.4) for φ_1 into Eq. (3.10) for I_{11}°, we obtain

$$I_{11}^\circ = \frac{1}{1+\sigma^2}\int_D [(\sigma b_{11} - b_{12} - 1)x_3^2 + (1+\sigma^2) b_{11} x_1 x_2 + (1+\sigma^2)(b_{12} - 1) x_2^2]\, dv$$

Integrating over the volume of the ellipsoid and substituting Expressions (5.7) for b_{jk} we find I_{11}° and analogously also the other components of tensor $\mathbf{I}^\circ$. After simplifications we obtain

$$I_{11}^\circ = -\frac{\gamma(a_1^2 + a_3^2)}{a_1^2 a_3^2 N}, \quad I_{22}^\circ = -\frac{\gamma(a_2^2 + a_3^2)}{a_2^2 a_3^2 N}, \quad I_{21}^\circ = -I_{12}^\circ = \frac{\gamma\sigma}{a_3^2 N}$$

$$I_{33}^\circ = -\frac{\gamma a_1^2 a_2^2}{a_1^2 + a_2^2}, \quad I_{3j}^\circ = I_{j3}^\circ = 0 \quad \left(\gamma = \frac{16\pi a_1 a_2 a_3}{15},\ j = 1,2\right) \tag{5.8}$$

Here N is determined by Eq. (5.7).

In case of irrotational flow ($\sigma = 0$) Eqs. (5.4) and (5.7) give

$$\varphi_1 = \varphi_1' = \frac{a_2^2 - a_3^2}{a_2^2 + a_3^2} x_2 x_3$$

and analogously also for φ_2 and φ_3, which coincides with the known expression of Zhukovskii's potentials for irrotational flow in an elliptical cavity [3].

For $\sigma = 0$ it follows from Eqs. (5.8) and (5.7) for N that

$$I_{11}^\circ = J_{11}'' - J_{11}', \qquad J_{11}'' = \frac{\gamma(a_2^2 - a_3^2)^2}{4(a_2^2 + a_3^2)}, \qquad J_{11}' = \frac{\gamma(a_2^2 + a_3^2)}{4} \tag{5.9}$$

Analogous equations (with cyclic transposition of indices) are valid for I_{22}° and I_{33}° when $\sigma = 0$, while the remaining components of tensor $\mathbf{I}^\circ$ in this case are equal to zero. Here J_{11}' and J_{11}'', respectively, are components of the tensor of inertia and the tensor of associated masses for the ellipsoid [3] with respect to point O in the case of density $\rho = 1$.

3. Let us examine a spherical cavity of radius a with center at the point O. Assuming that $a_1 = a_2 = a_3 = 0$ we obtain from Eqs. (5.4), (5.7) and (5.8) after simplification

$$\varphi_1 = \frac{\sigma(-2x_1 + \sigma x_2)x_3}{\sigma^2 + 4}, \qquad \varphi_2 = \frac{\sigma(-2x_2 - \sigma x_1)x_3}{\sigma^2 + 4}, \qquad \varphi_3 = \frac{\sigma(x_1^2 + x_2^2)}{2}$$

$$I_{11}^\circ = I_{22}^\circ = \frac{-4\gamma_0}{\sigma^2 + 4}, \quad I_{21}^\circ = -I_{12}^\circ = \frac{2\gamma_0\sigma}{\sigma^2 + 4}, \quad I_{33}^\circ = -\gamma_0 \quad \left(\gamma_0 = \frac{8\pi a^5}{15};\ j = 1, 2\right) \tag{5.10}$$

Functions φ_j' are obtained by means of substituting σ by $-\sigma$ in Eqs. (5.10) for φ_j.

4. Let us compute the tensor $\mathbf{I}^1$ for a spherical cavity of radius a. Into Eqs. (4.21) for I_{jk}^1 we substitute functions φ_j from (5.10), μ_1 and μ_2 from (4.7), $\mathbf{L}$ from (3.4) and components of the unit vector of the internal normal $\mathbf{n}$ to the surface of the sphere. Then we change to new variables ξ and ψ on the surface of the sphere according to Eqs.

$$x_1 = a\sqrt{1-\xi^2}\cos\psi, \qquad x_2 = a\sqrt{1-\xi^2}\sin\psi, \qquad x_3 = a\xi$$

After transformations and integration with respect to the angle ψ from 0 to 2π we obtain

$$I_{11}^1 = I_{22}^1 = -\frac{2\pi a^4}{(\sigma^2+4)^2}\int_{-1}^{1}\{(\mu_1^{-1}+\mu_2^{-1})[(7\sigma^2+4)\xi^2+4-\sigma^2]+$$
$$+i\sigma(\mu_1^{-1}-\mu_2^{-1})[(\sigma^2-4)\xi^3+(\sigma^2+4)\xi]\}\,d\xi$$
$$I_{12}^1 = -I_{21}^1 = -\frac{4\pi a^4}{(\sigma^2+4)^2}\int_{-1}^{1}\{\sigma(\mu_1^{-1}+\mu_2^{-1})[(\sigma^2-2)\xi^2+2]-$$
$$-i(\mu_1^{-1}-\mu_2^{-1})[2\sigma^2\xi^3+(\sigma^2+4)\xi]\}\,d\xi$$
$$I_{33}^1 = -\pi a^4\int_{-1}^{1}[(\mu_1^{-1}+\mu_2^{-1})(1-\xi^2)+i\sigma(\mu_1^{-1}-\mu_2^{-1})(\xi^3-\xi)]\,d\xi$$
$$\mu_{1,2} = \sqrt{2\omega_0}\sqrt{\sigma^{-1}\mp i\xi} \qquad (5.11)$$

Integrals (5.11) are evaluated in terms of elementary functions. After integration and reduction of similar terms we shall have

$$I_{11}^1 = I_{22}^1 = \varkappa[i(65\sigma^4+136\sigma^2-16)(\eta-\zeta)-\sigma(25\sigma^4-70\sigma^2-8)(\eta+\zeta)]$$
$$I_{12}^1 = -I_{21}^1 = \varkappa\sigma[i(21\sigma^4+6\sigma^2-56)(\eta-\zeta)+\sigma(93\sigma^2+28)(\eta+\zeta)]$$
$$I_{33}^1 = \varkappa(\sigma^2+4)^2[2i(4\sigma^2+1)(\eta-\zeta)+\sigma(5\sigma^2-1)(\eta+\zeta)]$$
$$\varkappa = -\frac{16\pi a^4}{105\sqrt{2\omega_0}\,\sigma^2(\sigma^2+4)^2} \qquad \eta=\sqrt{\sigma^{-1}-i}, \qquad \zeta=\sqrt{\sigma^{-1}+i} \qquad (5.12)$$

The remaining components of tensor $\mathbf{I}^1$ are equal to zero in accordance with (5.2). As radicals in Eqs. (5.12) those branches of the root are taken for which Re $\eta \leqslant 0$ and Re $\zeta \leqslant 0$. For $\sigma \to 0$ Eqs. (5.12) after removal of indeterminacies transform into corresponding equations of [8], taking into account differences in notation

6. Motion of body with fluid. With the assumptions made, the dynamics of a body with fluid are described by the last Eq. of (1.6) into which it is necessary to substitute $\mathbf{G}$ from (4.22). Let the tensors $\mathbf{I}^0$ and $\mathbf{I}^1$ be known for the given shape of cavity. Various formulations of problems on motion of a body with fluid are possible.

1. If the motion of the body is given (λ and Ω are known), then substituting the vector $\mathbf{G}$ from (4.22) into Eq. (1.6) we find the moment $\mathbf{M}$ which is necessary for maintaining of the given motion of the body with fluid.

2. Let the axis of rotation of the body O_1y_3, which goes through the center of inertia O_1 of the entire system parallel to axis Ox_3, be the main central axis of inertia of the system. Let us denote through J_{jk} the components of the tensor of inertia of the whole system $\mathbf{J}$ in the system of coordinates $O_1y_1y_2y_3$. The axes of these coordinates are parallel to the axes of system $Ox_1x_2x_3$(Fig. 1). Then we obtain Eqs.

$$J_{3j}=J_{j3}=0, \quad \mathbf{J}\cdot\boldsymbol{\omega}_0 = J_{33}\omega_0\mathbf{e}_3, \quad \boldsymbol{\omega}_0\times(\mathbf{J}\cdot\boldsymbol{\omega}_0)=0 \quad (j=1,2) \qquad (6.1)$$

It is assumed that the moment of external forces has the form $\mathbf{M}=\mathbf{M}_0e^{\lambda t}$, where the quantity λ and vector $\mathbf{M}_0$ are given. Calculation of forced oscillations of the system is reduced to determination of vector $\boldsymbol{\Omega}$. Substituting Eqs. (4.22) and (6.1) into the last Eq. of (1.6) we obtain

$$\lambda(\mathbf{J}+\rho\mathbf{I})\cdot\boldsymbol{\Omega}+\omega_0\mathbf{e}_3\times[(\mathbf{J}+\rho\mathbf{I})\cdot\boldsymbol{\Omega}-J_{33}\boldsymbol{\Omega}]=\mathbf{M}_0 \qquad (6.2)$$

In this manner the problem has been reduced to the solution of linear inhomogeneous Eq. (6.2) for vector $\boldsymbol{\Omega}$.

3. Let us examine the more complicated and interesting problem of natural oscillations of a rotating body with a fluid near a steady rotation. The moment of external forces with respect to point O_1 is taken as equal to zero ($\mathbf{M}=0$, the body is free) and let, as previously, the axis of rotation O_1y_3 be the main axis of inertia of the system, i.e. Eqs. (6.1) are satisfied. We substitute $\mathbf{M}_0=0$ and Eq. (3.4) for σ into Eq. (6.2), assuming that $\lambda\neq 0$

$$(\mathbf{J}+\rho\mathbf{I})\cdot\boldsymbol{\Omega}+{}^1\!/\!_2\sigma\mathbf{e}_3\times[(\mathbf{J}+\rho\mathbf{I})\cdot\boldsymbol{\Omega}-J_{33}\boldsymbol{\Omega}]=0 \qquad (6.3)$$

The determinant of the linear homogeneous (with respect to $\boldsymbol{\Omega}$) system (6.3) is set equal to zero and taking into account (6.1) and (4.22) we obtain the characteristic equation

$$\begin{vmatrix} K_{11}-{}^1\!/\!_2\sigma K_{21}, & K_{12}-{}^1\!/\!_2\sigma(K_{22}-J_{33}), & \rho(I_{13}-{}^1\!/\!_2\sigma I_{23}) \\ K_{21}+{}^1\!/\!_2\sigma(K_{11}-J_{33}), & K_{22}+{}^1\!/\!_2\sigma K_{12}, & \rho(I_{23}+{}^1\!/\!_2\sigma I_{13}) \\ \rho I_{31}, & \rho I_{32}, & J_{33}+\rho I_{33} \end{vmatrix} = f(\sigma,\rho,\sqrt{\nu})=0$$

$$K_{jk} = J_{jk} + \rho I_{jk}, \quad I_{jk} = I_{jk}^{\circ} + \sqrt{\nu}\, I_{jk}^{1}, \quad J_{jk} = J_{kj} \quad (j, k = 1, 2, 3) \quad (6.4)$$

Here the components J_{jk} are constant while I_{jk}° and I_{jk}^{1} are functions of σ depending on the shape of the cavity. The roots σ of Eq. (6.4) determine the eigen numbers $\lambda = 2\omega_0/\sigma$ of the problem on oscillations of a rotating body with fluid. Let us examine Eq. (6.4) in some cases.

4. Let the ratio of the mass of fluid to the mass of the entire system be small, i.e. $\rho \ll 1$ (see end of Section 1). In the case of a solid body without fluid ($\rho = 0$) Eq. (6.4) is reduced to a quadratic Eq.

$$[(J_{33} - J_{11})\ (J_{33} - J_{22}) - J_{12}^2]\ \sigma^2 + 4\ (J_{11}J_{22} - J_{12}^2) = 0 \quad (6.5)$$

The free term of (6.5) is positive since $\mathbf{J}$ is a positive definite tensor. For stability of rotation it is necessary that σ be purely imaginary and for this it is required that

$$(J_{33} - J_{11})\ (J_{33} - J_{22}) \geqslant J_{12}^2 \quad (6.6)$$

This is a well-known condition for stability of stationary rotation of a free solid body. Without destroying generality the principal central axes of inertia of the system are selected to be axes y_1 and y_2. Then $J_{12} = 0$ and condition (6.6) are reduced to the requirement that the moment of inertia J_{33} be either the largest or the smallest principal central moment of inertia of the system. The roots of Eq. (6.5) for $J_{12} = 0$ are:

$$\sigma_{1,2}^{\circ} = \pm 2i\ (J_{11}J_{22})^{1/2}\ [(J_{33} - J_{11})\ (J_{33} - J_{22})]^{-1/2} \quad (6.7)$$

The roots of Eq. (6.4) for $\rho \ll 1$ are determined by the perturbation method. We assume in Eq. (6.4)

$$\sigma_s = \sigma_s^{\circ} + \rho\delta_s^{\circ} \qquad (s = 1,\ 2) \quad (6.8)$$

Taking into account that σ_s° is a root of the function $f(\sigma_s^{\circ}, 0, \sqrt{\nu})$, we obtain from (6.4), with accuracy to small terms of higher order,

$$\delta_s^{\circ} = -\frac{\partial f / \partial \rho}{\partial f / \partial \sigma} \quad \text{for} \quad \sigma = \sigma_s^{\circ},\ \rho = 0 \quad (6.9)$$

Derivatives in Eq. (6.9) are computed according to rules of differentiation of determinant (6.4) assuming that $J_{12} = 0$. Subsequently σ_s° is substituted from Eq. (6.7). We obtain corrections to σ in the form

$$\delta_s^{\circ} = \frac{J_{33}}{(J_{33} - J_{11})\,(J_{33} - J_{22})} \left[\frac{2}{\sigma_s^{\circ}} \left(\frac{I_{11}J_{22}}{J_{11} - J_{33}} + \frac{I_{22}J_{11}}{J_{11} - J_{33}}\right) + I_{21} - I_{12}\right] \quad (s=1,2) \quad (6.10)$$

Values of I_{jk} in (6.10) must be taken for $\sigma = \sigma_s^{\circ}$.

For an ideal fluid ($\nu = 0$) it is necessary in Eq. (6.10) to take $I_{jk} = I_{jk}^{\circ}$. Let the stability condition (6.6) be fulfilled, and both roots σ_s° from (6.7) be purely imaginary quantities. As follows from Eq. (3.16) in this case, I_{11}° and I_{22}° will be real, while the difference $I_{21}^{\circ} - I_{12}^{\circ}$ will be a purely imaginary quantity. Consequently, corrections δ_s° from (6.10) will be purely imaginary in this case. If, however, the quantities σ_s° are real, then all components of the tensor $\mathbf{I}^{\circ}$ are also real (see Section 3) and corrections δ_s° turn out to be real. In this manner the presence of a small mass of ideal fluid in the cavity of a rotating solid body in the first approximation with respect to parameter ρ, does not change the stability (or lack of stability) of motion of the solid body leaving the roots of the characteristic equation purely imaginary (or real).

5. Let us examine Eq. (6.4) in the case where the fluid is ideal ($\nu = 0$) and the cavity has the shape of an ellipsoid (5.3). Substituting Formulas (5.8) into Eq. (6.4) (after expansion of the determinant and some transformations) we obtain

$$A_1\sigma^4 + A_2\sigma^2 + A_3 = 0$$

$$A_1 = (J_{33} - J_{11})\ (J_{33} - J_{22}) - J_{12}^2,\ A_2 = (a_1^2 + a_3^2)\ (a_2^2 + a_3^2)\ a_1^{-2}a_2^{-2} \times$$

$$\times [(J_{33} - J_{11}^{\circ})\ (J_{33} - J_{22}^{\circ}) - J_{12}^2] + 4\ (J_{11}J_{22} - J_{12}^2) - 4\rho\gamma a_3^2 J_{33}$$

$$A_3 = 4\ (a_1^2 + a_3^2)\ (a_2^2 + a_3^2)\ a_1^{-2}\ a_2^{-2}(J_{11}^{\circ}J_{22}^{\circ} - J_{12}^2)$$

$$J_{11}^{\circ} = J_{11} - \rho\gamma a_2^2 a_3^2\ (a_2^2 + a_3^2)^{-1}, \qquad J_{22}^{\circ} = J_{22} - \rho\gamma a_1^2 a_3^2\ (a_1^2 + a_3^2)^{-1} \quad (6.11)$$

Here γ is determined from (5.8). Quantity J_{11}° (and analogously J_{22}°) are presented in the form

$$J_{11}^{\circ} = J_{11} - \rho\ (J_{11}' - J_{11}'')$$

where J_{11}' and J_{11}'' are introduced in (5.9). Quantities J_{jk}° are components (in axes $O_1y_1y_2y_3$) of the tensor of inertia $\mathbf{J}^\circ$ of that solid body which has an equivalent body with a cavity filled with ideal fluid ~~in case~~ of irrotational flow [3]. We note that from the positive definiteness of tensor $\mathbf{J}^\circ$ and Eq. $J_{12}^\circ = J_{12}$ it follows that $A_3 \geqslant 0$. It is easy to find σ from Eq. (6.11) and subsequently also $\lambda = 2\omega_0/\sigma$. For stability it is necessary that all roots of Eq. (6.11) be purely imaginary (in this paper stability is examined in the linear approximation). For this it is easy to see that

$$A_1 \geqslant 0, \qquad A_2 \geqslant 2\sqrt{A_1A_3}$$

is necessary and sufficient (for $A_3 \geqslant 0$).

Taking into account Expression (6.11) for A_1, the first of these conditions of stability coincides with inequality (6.6) and is the stability condition for rotation of a free solid body which is obtained when all fluid solidifies in the cavity.

The motion of a body with an ellipsoidal cavity filled with ideal fluid has been investigated by many authors using different methods (for example, [1 to 3]). An equation analogous to (6.11) was derived and in a number of cases analyzed in [11].

Let, in particular, the system have dynamic symmetry ($J_{11} = J_{22}$, $J_{12} = 0$) and the cavity be an ellipsoid of rotation ($a_1 = a_2$). Then it is easy to verify that the biquadratic Eq. (6.11) can be presented in the form

$$A_1\sigma^4 + A_2\sigma^2 + A_3 = (A_4\sigma^2 + iA_5\sigma + A_6)\ (A_4\sigma^2 - iA_5\sigma + A_6) = 0$$

$$A_4 = J_{33} - J_{11}, \qquad A_5 = (a_1^2 + a_3^2)\ a_1^{-2}(J_{33} - J_{11}^\circ) - 2J_{11}$$

$$A_6 = 2\,(a_1^2 + a_3^2)a_1^{-2}J_{11}^\circ \tag{6.12}$$

In [1 and 2] the characteristic equation was obtained for a symmetrical top with an axisymmetric ellipsoidal cavity filled with an ideal fluid. If in equations of papers [1 and 2] the gravity force is set equal to zero, quadratic equations are obtained which can be shown to be equivalent to equations into which Eq. (6.12) is expanded. For stability it is necessary and sufficient that $A_5^2 + 4A_4A_6 \geqslant 0$.

6. In the case of fluid with low viscosity ($\nu \ll 1$) we apply the method of perturbations to Eq. (6.4). Let σ' be some nonmultiple root of (6.4) for an ideal fluid (for $\nu = 0$). The root σ of (6.4) which for $\nu \ll 1$ is close to σ' and the corresponding characteristic value is sought in the form

$$\sigma = \sigma' + \nu^{1/2}\delta, \qquad \lambda = 2\omega_0/\sigma = 2\omega_0\ [(\sigma')^{-1} - \nu^{1/2}(\sigma')^{-2}\ \delta] \tag{6.13}$$

Substituting σ from (6.13) into Eq. (6.4) and taking into account that $f(\sigma', \rho, 0) = 0$ we obtain, with accuracy to small terms of higher order, in analogy to (6.9)

$$\delta = -\frac{\partial f / \partial \sqrt{\nu}}{\partial f / \partial \sigma} \qquad \text{for}\quad \sigma = \sigma', \quad \nu = 0 \tag{6.14}$$

Actual computations are carried out for a spherical cavity of radius a. Without destroying generality it is assumed that for $j = 1, 2, 3$ the axes O_1y_j are the principal central axes of the inertial system. Moments of inertia (6.11) of an equivalent solid body in the given case are

$$[illegible]\ 2r^1 \qquad J_{jk}^\circ = J_{jk} = 0 \qquad (j \neq k,\ j,\ k = 1, 2, 3) \tag{6.15}$$

Here γ_0 is determined by Eq. (5.10). Tensor $\mathbf{J}^\circ$, as determined by Eq. (6.15), is the tensor of the inertial system in which all the fluid is replaced by a point mass equal to the mass of fluid and located at the center O of the sphere.

We note identities which follow from Eqs. (5.10)

$$I_{11}^\circ - {}^1/_2\sigma I_{21}^\circ = I_{22}^\circ + {}^1/_2\sigma I_{12}^\circ = -\gamma_0, \qquad I_{12}^\circ - {}^1/_2\sigma I_{22}^\circ = I_{21}^\circ + {}^1/_2\sigma I_{11}^\circ = 0$$

These identities and also Eqs. (5.2) and (5.10) and the expression for J_{jk} in terms of J_{jk}° according to (6.15) are substituted into determinant (6.4). We obtain

$$f(\sigma, \rho, \sqrt{\nu}) = (J_{33}^\circ + \rho\sqrt{\nu}\, I_{33}^1) \times \tag{6.16}$$

$$\times \begin{vmatrix} J_{11}^\circ + \rho\sqrt{\nu}\,(I_{11}^1 - {}^1/_2\,\sigma I_{21}^1) & {}^1/_2\sigma\,(J_{33}^\circ - J_{22}^\circ) + \rho\sqrt{\nu}\,(I_{12}^1 - {}^1/_2\sigma I_{22}^1) \\ {}^1/_2\sigma\,(J_{11}^\circ - J_{33}^\circ) + \rho\sqrt{\nu}\,(I_{21}^1 + {}^1/_2\sigma I_{11}^1), & J_{22}^\circ + \rho\sqrt{\nu}\,(I_{22}^1 + {}^1/_2\,\sigma I_{21}^1) \end{vmatrix} = 0$$

Assuming $\nu = 0$ in Eq. (6.16) we obtain an Eq. of the form (6.5) with exchange of J_{jk} for J_{jk}°. In this manner a body with a spherical cavity filled with an ideal fluid is equivalent to a solid body with a tensor of inertia $\mathbf{J}^\circ$. Computing derivatives of f from (6.16) we also find

δ from Eq. (6.14). In analogy to (6.7) and (6.10) we obtain

$$\sigma'_{1,2} = \pm ik, \qquad k = 2\,(J_{11}^\circ J_{22}^\circ)^{1/2}\,[(J_{33}^\circ - J_{11}^\circ)(J_{33}^\circ - J_{22}^\circ)]^{-1/2}$$

$$\delta = \frac{\rho J_{33}^\circ}{(J_{33}^\circ - J_{11}^\circ)(J_{33}^\circ - J_{22}^\circ)}\left[\frac{2}{\sigma'}\left(\frac{I_{11}^1 J_{22}^\circ}{J_{11}^\circ - J_{33}^\circ} + \frac{I_{22}^1 J_{11}^\circ}{J_{22}^\circ - J_{33}^\circ}\right) + I_{21}^1 - I_{12}^1\right] \tag{6.17}$$

Eqs. (6.13), (6.17) and (5.12) determine eigen numbers of the problem under examination If J_{33}° is the intermediate in magnitude, moment of inertia (for example $J_{11}^\circ < J_{33}^\circ < J_{22}^\circ$) then both roots σ' are real and the motion of the body is unstable in the absence of viscosity. A low viscosity will change these roots somewhat in accordance with Eqs. (6.13), however, generally speaking, the motion will remain unstable.

More interesting is the case when J_{33}° is the largest or smallest principal central moment of inertia of the equivalent body. Then k from (6.17) is real. From the known inequality $J_{33}^\circ \leqslant J_{11}^\circ + J_{22}^\circ$ for moments of inertia, the following inequality results

$$J_{11}^\circ J_{22}^\circ \geqslant (J_{33}^\circ - J_{11}^\circ)\,(J_{33}^\circ - J_{22}^\circ) > 0$$

From this we obtain on the basis of (6.17) that $k \geqslant 2$. Radicals in Eqs. (5.12) are computed selecting branches with negative real parts.

$$\eta = -(1-i)\,\sqrt{k \pm 1}\,/\,\sqrt{2k}, \quad \zeta = -(1+i)\,\sqrt{k \mp 1}\,/\,\sqrt{2k} \quad (\sigma' = \pm ik) \tag{6.18}$$

Here and in the following the upper and lower signs correspond to the selection of signs in Eq. (6.17) for σ'. Substituting Eqs. (5.12) and (6.18) into Eqs. (6.13) and (6.17) we obtain

$$\sigma = \pm ik + \sqrt{\nu}\;\delta, \qquad \lambda = 2\omega_0\,(\mp ik^{-1} + \sqrt{\nu}\;k^{-2}\delta)$$

$$\delta = \frac{\rho\pi a^4\,(J_{33}^\circ - J_{11}^\circ)^2\,(J_{33}^\circ - J_{22}^\circ)^2\,(B_1 \pm iB_2)}{420 J_{11}^\circ J_{22}^\circ J_{33}^\circ\,\sqrt{k\omega_0}\,k\,(J_{11}^\circ + J_{22}^\circ - J_{33}^\circ)^2}$$

$$B_1 = b\,(h_1 g_1 - h_2 g_2) - k\,(h_3 g_2 - h_4 g_1), \qquad B_2 = b\,(h_2 g_1 - h_1 g_2) + k\,(h_3 g_1 - h_4 g_2)$$

$$b = J_{22}^\circ\,/\,(J_{11}^\circ - J_{33}^\circ) + J_{11}^\circ\,/\,(J_{22}^\circ - J_{33}^\circ), \qquad h_1 = 25k^5 + 70\,k^3 - 8k$$

$$h_2 = 65\,k^4 - 136\,k^2 - 16, \quad h_3 = 21\,k^5 - 6k^3 - 56\,k, \quad h_4 = 93\,k^3 - 28$$

$$g_1 = \sqrt{k+1} + \sqrt{k-1}, \; g_2 = \sqrt{k+1} - \sqrt{k-1} \quad (k \geqslant 2) \tag{6.19}$$

From Eqs. (6.19) for δ and λ it follows that both eigen numbers λ for the body with viscous fluid have the same real parts while their imaginary parts differ in sign. The stability of motion is determined by the sign of the real part δ_j, i.e. by the sign of quantity B_1.

In order to evaluate the sign of B_1 some supplementary inequalities are obtained. Since the function $g_1(k)$ from (6.19) is monotonously increasing, while $g_2(k)$ monotonously decreases, we have for $k \geqslant 2$

$$g_1(k) \geqslant g_1(2) = \sqrt{3} + 1 > 2.7, \qquad g_2(k) \leqslant g_2(2) = \sqrt{3} - 1 < 0.8 \tag{6.20}$$

An upper estimate is also obtained for functions g_1 and simple estimates are made for function h_j for $k \geqslant 2$ from (6.19)

$$g_1^2 = 2k + 2\sqrt{k^2 - 1} < 4k \leqslant 2k^2, \qquad g_1 < \sqrt{2}\,k < 1.5\,k$$

$$h_1 > 25\,k^5, \quad h_2 < 65\,k^4, \quad 0 < h_3 < 21\,k^5, \quad 0 < h_4 < 93k^3 < 47\,k^3 \tag{6.21}$$

From inequalities (6.20) and (6.21) for $k \geqslant 2$ the following inequalities are obtained

$$h_1 g_1 - h_2 g_2 > k^4\,(25 \cdot 2.7k - 65 \cdot 0.8) > k^4\,(67\,k - 52)$$

$$h_3 g_2 - h_4 g_1 < 21 \cdot 0.8\;k^5 < 17\,k^5$$

$$h_3 g_2 - h_4 g_1 > -47 \cdot 1.5\;k^4 > -71\,k^4 \tag{6.22}$$

Combining the first inequality (6.22) at first with the second and then with the third inequality of (6.22) we obtain

$$\begin{aligned}(h_1 g_1 - h_2 g_2) - (h_3 g_2 - h_4 g_1) &> k^4\,(50\,k - 52) > 0\\ (h_1 g_1 - h_2 g_2) + (h_3 g_2 - h_4 g_1) &> k^4\,(67\,k - 123) > 0\end{aligned} \qquad (k \geqslant 2)$$

From here

$$h_1 g_1 - h_2 g_2 > |\,h_3 g_2 - h_4 g_1\,| > 0 \tag{6.23}$$

At first, let us examine the case where J_{33}° will be the smallest moment of inertia: $J_{33}^\circ < J_{11}^\circ$, $J_{33}^\circ < J_{22}^\circ$. It follows then from the inequality between the arithmetic and geometric mean (k is determined from Eq. (6.17)) that

$$b = J_{22}^\circ (J_{11}^\circ - J_{33}^\circ)^{-1} + J_{11}^\circ (J_{22}^\circ - J_{33}^\circ)^{-1} \geqslant k \tag{6.24}$$

From inequalities (6.23) and (6.24) and Eq. (6.19) for B_1 it follows that

$$B_1 \geqslant k\ [(h_1g_1 - h_2g_2) - (h_3g_2 - h_4g_1)] > 0$$

i.e. the motion is unstable.

Let $J_{33}^\circ > J_{11}^\circ$, $J_{33}^\circ > J_{22}^\circ$, i.e. J_{33}° is the largest among the moments of inertia. In this case instead of (6.24) we have the inequality

$$b = J_{22}^\circ (J_{11}^\circ - J_{33}^\circ)^{-1} + J_{11}^\circ (J_{22}^\circ - J_{33}^\circ)^{-1} \leqslant -k < 0 \tag{6.25}$$

From inequalities (6.23) and (6.25) it follows that

$$B_1 \leqslant -k\ [(h_1g_1 - h_2g_2) + (h_3g_2 - h_4g_1)] < 0$$

i.e. the motion is stable. In this manner the rotation around the axis O_1y_3 of a free solid body with spherical cavity filled with a viscous fluid is stable if $J_{33}^\circ > J_{11}^\circ$ and $J_{33}^\circ > J_{22}^\circ$, it is unstable if in even one of these two conditions the inequality sign is exchanged for the opposite one. Instability in this case is connected with the viscosity of the fluid and is absent for $\nu = 0$. From Eqs. (6.15) it follows that stability conditions can be written using moments of inertia of the entire system in the form $J_{33} > J_{11}$ and $J_{33} > J_{22}$ which coincides with known results [3]. We note that above, not only the conditions for stability were obtained but also the roots of the characteristic equation, in particular the decrements of damping, were computed.

The author thanks D.E. Okhotsimskii for discussion and formulation of the problem.

BIBLIOGRAPHY

1. Sobolev, S.L., On the motion of a symmetrical top with a fluid-filled cavity. PMTF, No. 3, 1960.
2. Ishlinskii, A.Iu. and Temchenko, M.E., On small oscillations of the vertical axis of a top which has a cavity completely filled with an ideal incompressible fluid. PMTF, No. 3, 1960.
3. Moiseev, N.N. and Rumiantsev, V.V., Dynamics of a body with cavities containing fluid. M., Izd. "Nauka", 1965.
4. Stewartson, K. and Roberts, P.H., On the motion of a liquid in a spheroidal cavity of a precessing rigid body. J. Fluid Mech., Vol. 17, 1963.
5. Greenspan, H.P., On the transient motion of a contained rotating fluid. J. Fluid Mech., Vol. 20, 1964.
6. Greenspan, H.P., On the general theory of contained rotating fluid motions. J. Fluid Mech., Vol. 22, 1965.
7. Chernous'ko, F.L., Motion of a solid body with cavities filled with viscous fluid at small Reynolds numbers. Zh. Vychisl. Mat. mat. Fiz., Vol. 5, No. 6, 1965.
8. Chernous'ko, F.L., Motion of a body with a cavity filled with a viscous fluid at large Reynolds numbers. PMM, Vol. 30, No. 3, 1966.
9. Bykhovskii, E.B. and Smirnov, N.V., On orthogonal expansion of the space of vector functions which are quadratically summable over the given region, and on operators of vector analysis. Trudy Mat. Inst. im. V.A. Steklova Akad. Nauk. SSSR, Vol. 59, 1960.
10. Norden, A.P., Theory of Surfaces. M., Gostekhizdat, 1956.
11. Hough, S.S., The oscillations of a rotating ellipsoidal shell containing fluid. Philos. Trans. Roy. Soc., London, A., Vol. 186, 1895.

Translated by B. D.